REMARK ON THE ALEXANDER POLYNOMIALS OF PERIODIC KNOTS

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To Yuko's 29th birthday

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1. Introduction

Let p be a prime number and K a non-trivial knot in S^3 which has period p. Then the Alexander polynomial $\Delta_K(t)$ of such a knot must have some distinguished properties as Murasugi [3] have revealed.

In this note, we shall show that under a certain assumption on the Alexander polynomial $\Delta_K(t)$, it is uniquely determined only by p.

The proof will be done by applying number theory to Murasugi's work [3] on a necessary condition on $\Delta_K(t)$ for periodic knots K.

2. Result

For a knot K in S^3, we denote by $\Delta_K(t) \in \mathbb{Z}[t]$ the Alexander polynomial of K normalized such that $\Delta_K(0) \neq 0$ and the leading coefficient of it is positive.

Our main result is the following:

Theorem 1. Let p be an odd prime number and $K \subseteq S^3$ a non-trivial periodic knot of period p. If $\Delta_K(t)$ is monic and has degree $p-1$, then we have

$$\Delta_K(t) = \sum_{n=0}^{p-1} (-1)^n t^n = t^{p-1} - t^{p-2} + \cdots - t + 1.$$

For the case where degree $p-1$ Alexander polynomials of p-periodic knots have general leading coefficients, we will show the following:

Theorem 2. Let p be an odd prime number and define $\Pi(p)$ to be the set of all the p-periodic knots in S^3 whose Alexander polynomial has degree $p-1$. Also, for a finite set S of prime numbers which does not

contain p, we define the set $\mathcal{D}(p,S) \subseteq \mathbb{Z}[t]$ to be the collection of all the $\Delta_K(t)$'s such that $K \in \Pi(p)$ and the leading coefficient of $\Delta_K(t)$ is prime to the prime numbers outside S. Then $\mathcal{D}(p,S)$ is finite and

$$\#\mathcal{D}(p,S) \leq \left(\frac{p+1}{2}\right)^{3\cdot 7^{\frac{3(p-1)}{2}+\#S(\mathbb{Q}(\zeta_p))}},$$

where $S(\mathbb{Q}(\zeta_p))$ denotes the set of all the primes of the p-th cyclotomic field $\mathbb{Q}(\zeta_p)$ lying over the prime numbers in S.

Remark 1. (1) If K is fibred, then $\Delta_K(t)$ is monic.
(2) For a non-trivial knot K of prime period p, if the leading coefficient of $\Delta_K(t)$ is prime to p, then $\deg \Delta_K(t) \geq p-1$ (See Davis and Livingston[2, Cor. 4.2]).

3. Proof of Theorem 1.

Let T be a transformation of S^3 of order p such that $T(K) = K$ and it acts on K fixed point freely, and we denote by $B \subseteq S^3$ the set of the fix points of T. Then B is the unknot and the quotient space S^3/T is homeomorphic to S^3, and we let $\overline{K}$ and $\overline{B}$ be the quotient knots of K and B in S^3/T, respectively. We write for $\Delta_{\overline{K}\cup\overline{B}}(t,u) \in \mathbb{Z}[t,u]$ the two-variable Alexander polynomial of the link $\overline{K}\cup\overline{B}$ with $\Delta_{\overline{K}\cup\overline{B}}(t,u) \notin t\mathbb{Z}[t,u] \cup u\mathbb{Z}[t,u]$ (Note that $\Delta_{\overline{K}\cup\overline{B}}(t,u)$ is defined up to ± 1). We put λ the linking number of K and B.

It follows from Murasugi [3] that

$$\Delta_K(t) = \Delta_{\overline{K}}(t) \prod_{i=1}^{p-1} \Delta_{\overline{K}\cup\overline{B}}(t,\zeta_p^i) \tag{1}$$

with a primitive p-th root of unity ζ_p. Also, by using Murasugi's congruence [3],

$$\Delta_K(t) \equiv \pm t^j \left(\frac{t^\lambda - 1}{t-1}\right)^{p-1} \Delta_{\overline{K}}(t)^p \pmod p$$

for some $j \in \mathbb{Z}$, we derive $\lambda = 2$ and $\Delta_{\overline{K}}(t) = 1$, because $\Delta_{\overline{K}}(t) \mid \Delta_K(t)$ in $\mathbb{Z}[t]$ by (1), $\deg \Delta_K(t) = p-1$, and the leading coefficient of $\Delta_K(t)$ is prime to p. Hence we may assume that

$$\Delta_{\overline{K}\cup\overline{B}}(t,\zeta_p^i) = g(\zeta_p^i)t - h(\zeta_p^i) \tag{2}$$

with some $g(u), h(u) \in \mathbb{Z}[u]$ for $1 \leq i \leq p-1$ by (1).

Because $\Delta_K(t)$ is monic, we see that $\eta_1 := g(\zeta_p)$ is a unit of the ring $\mathbb{Z}[\zeta_p]$ by the relation $\prod_{i=1}^{p-1} g(\zeta_p^i) = 1$, which comes from (1). Also, since

the constant term of $\Delta_K(t)$ is equal to 1, we find that $\eta_2 := h(\zeta_p)$ is a unit of $\mathbb{Z}[\zeta_p]$. Hence if we put $\varepsilon := \eta_1^{-1}\eta_2 \in \mathbb{Z}[\zeta_p]^\times$, then we have

$$(3)\quad \Delta_K(t) = \prod_{i=1}^{p-1} g(\zeta_p^i) \prod_{i=1}^{p-1}(t - g(\zeta_p^i)^{-1}h(\zeta_p^i)) = \prod_{\sigma\in \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})} (t-\sigma(\varepsilon)).$$

On the other hand, it follows from the second Torres condition (See [4]) that

$$(4)\qquad \Delta_{\overline{K}\cup\overline{B}}(t^{-1}, \zeta_p^{-1}) = t^a\zeta_p^b\Delta_{\overline{K}\cup\overline{B}}(t, \zeta_p)$$

for some $a, b \in \mathbb{Z}$. Then we find from (2) that $a = -1$ and

$$h(\zeta_p^{-1}) = -\zeta_p^b g(\zeta_p),\ \ g(\zeta_p^{-1}) = -\zeta_p^b h(\zeta_p).$$

Therefore, if we denote by $J \in \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ the complex conjugation, we have

$$(5)\quad J(\varepsilon) = J(h(\zeta_p))/J(g(\zeta_p)) = h(\zeta_p^{-1})/g(\zeta_p^{-1}) = g(\zeta_p)/h(\zeta_p) = \varepsilon^{-1}.$$

We need the following fact from the theory of cyclotomic fields (see [5, Prop.1.5]):

Lemma 1. For any $\varepsilon \in \mathbb{Z}[\zeta_p]^\times$, there exist $r \in \mathbb{Z}$ and $\varepsilon_0 \in \mathbb{Z}[\zeta_p+\zeta_p^{-1}]^\times$ such that

$$\varepsilon = \zeta_p^r\varepsilon_0.$$

By Lemma 1 and (5), we obtain

$$\varepsilon^{-1} = J(\varepsilon) = J(\zeta_p)^r J(\varepsilon_0) = \zeta_p^{-r}\varepsilon_0 = \zeta_p^{-2r}\varepsilon,$$

from which we derive

$$\varepsilon = \pm\zeta_p^r.$$

Because $\Delta_K(1) = \pm 1$ and $\Delta_K(-1) \neq 0$, we conclude that $\varepsilon = -\zeta_p^r$ with some $r \in \mathbb{Z}$ prime to p, and

$$\Delta_K(t) = t^{p-1} - t^{p-2} + \cdots - t + 1$$

by (3). Thus we have proved Theorem 1. □

4. Proof of Theorem 2

We will give the following proposition, from which we can easily derive Theorem 2:

Proposition 1. For a finite extension field $F/\mathbb{Q}$, a finite set S of prime numbers, and positive integer m, we define the set $\mathcal{P}(F, S, m) \subseteq \mathbb{Z}[t]$ to be the collection of $\Delta_K(t)$'s for the knots K in S^3 such that the leading coefficient of $\Delta_K(t)$ is prime to the prime numbers outside S, the splitting field $\mathrm{Spl}(\Delta_K(t))$ of $\Delta_K(t)$ over $\mathbb{Q}$ is contained in F, and

the multiplicity of each zero of $\Delta_K(t)$ is at most m. Then $\mathcal{P}(F,S,m)$ is finite and we have

$$\#\mathcal{P}(F,S,m) \leq (m+1)^{3\cdot 7^{[F:K]+\#(S(F)\cup\infty(F))}},$$

where $S(F)$ and $\infty(F)$ denote the set of the primes of F lying over the prime numbers in S and that of the archimedean primes of F, respectively.

Proof. Let $\Delta_K(t) \in \mathcal{P}(F,S,m)$. Then we have

$$\Delta_K(t) = a\prod_{i=1}^{d}(X-\alpha_i)^{m_i} \in \mathbb{Z}[t]$$

for some $a \in \mathbb{Z}$ which is prime to the prime numbers outside S, $0 \leq d \in \mathbb{Z}$, distinct α_i's in F, and $1 \leq m_i \leq m$.

Let $\mathfrak{p} \notin S(F)$ be any non-archimedean prime of F and $v_{\mathfrak{p}}$ a $\mathfrak{p}$-adic valuation of F. Since $a\alpha_i$ is integral over $\mathbb{Z}$ and $v_{\mathfrak{p}}(a) = 0$, we see

$$v_{\mathfrak{p}}(\alpha_i) = v_{\mathfrak{p}}(a\alpha_i) \geq 0.$$

Because the constant term and the leading coefficient of $\Delta_K(t)$ are coincide, we have

$$a = \pm\prod_{i=1}^{d}\alpha_i^{m_i},$$

from which we derive

$$0 = v_{\mathfrak{p}}(a) = \sum_{i=1}^{d} m_i v_{\mathfrak{p}}(\alpha_i).$$

Therefore we find that $v_{\mathfrak{p}}(\alpha_i) = 0$ for $1 \leq i \leq d$, which means that α_i's are $S(F)$-units of F.

On the other hand, $a(1-\alpha_i) = a - a\alpha_i$ is also integral over $\mathbb{Z}$, we obtain

$$v_{\mathfrak{p}}(1-\alpha_i) = v_{\mathfrak{p}}(a(1-\alpha_i)) \geq 0.$$

Also, since $\Delta_K(1) = \pm 1$, we have

$$a\prod_{i=1}^{d}(1-\alpha_i)^{m_i} = \pm 1,$$

from which we derive

$$0 = v_{\mathfrak{p}}(\pm 1) = v_{\mathfrak{p}}(a) + \sum_{i=1}^{d} m_i v_{\mathfrak{p}}(1-\alpha_i) = \sum_{i=1}^{d} m_i v_{\mathfrak{p}}(1-\alpha_i).$$

Hence $v_{\mathfrak{p}}(1-\alpha_i) = 0$ for $1 \leq i \leq d$, which means that $1-\alpha_i$'s are also $S(F)$-units of F.

Now we apply the following result from analytic number theory given by Evertse [1]:

Lemma 2. Let F be a finite extension of $\mathbb{Q}$ and T a finite set of non-archimedean primes of F. Then the number of solutions (X, Y) of the equation

$$X + Y = 1$$

in the T-unit group of F is at most $3 \cdot 7^{[F:\mathbb{Q}]+\#(T\cup\infty_F)}$.

As we have seen in the above, $(\alpha_i, 1-\alpha_i)$ is a solution in the $S(F)$-unit group of F of the equation $X + Y = 1$. Hence, it follows from Lemma 2 that the number of such α_i's is at most $3 \cdot 7^{[F:\mathbb{Q}]+\#(T\cup\infty_F)}$. Therefore we obtain

$$\#\mathcal{P}(F, S, m) \leq (m+1)^{3\cdot 7^{[F:\mathbb{Q}]+\#(T\cup\infty_F)}}.$$

□

Now we will derive Theorem 2 from Proposition 1. Assume $K \in \mathcal{D}(p, S)$. Then, as the proof of Theorem 1, we find that

$$\Delta_K(t) = \prod_{i=1}^{p-1}(g(\zeta_p)t - h(\zeta_p))$$

for some $g(u), h(u) \in \mathbb{Z}[u]$, since the leading coefficient of $\Delta_K(t)$ is prime to p by $p \notin S$ and $\deg \Delta_K(t) = p-1$. Hence $\mathrm{Spl}(\Delta_K(t)) \subseteq \mathbb{Q}(\zeta_p)$ and $\Delta_K(t) \in \mathcal{P}(\mathbb{Q}(\zeta_p), S, \frac{p-1}{2})$ because $\Delta_K(t)$ has at least two distinct zeros. Therefore, applying Proposition 1, we complete the proof of Theorem 2 by using the facts $[\mathbb{Q}(\zeta_p) : \mathbb{Q}] = p-1$ and $\#\infty_{\mathbb{Q}(\zeta_p)} = \frac{p-1}{2}$.
□

References

[1] J.-H. Evertse, On equations in S-units and the Thue-Mahler equation, Invent. Math. **75**, 561–584.

[2] J. Davis, C. Livingston, Alexander polynomials of periodic knots, Topology **30** (1991), no. 4, 551–564.

[3] K. Murasugi, On periodic knots, Comment. Math. Helv. **46** (1971), 162–174.

[4] G. Torres, On the Alexander polynomial, Ann. of Math. **57** (1953), 57–89.

[5] L.C. Washington, Introduction to cyclotomic fields (2nd ed.), Graduate Texts in Mathematics **83** (1997), New York, Springer.